$\mathbb{Q}$-factoriality of double solids

A variety X is called $\mathbb{Q}$-factorial if a multiple of each Weil divisor of X is Cartier. The $\mathbb{Q}$-factoriality is a very subtle property. It depends on both local types of singularities and their global position. Also it depends on the field of definition of the variety. In this talk, we are interested in the $\mathbb{Q}$-factoriality of a double solid defined over $\mathbb{C}$, *i.e*, a double cover of $\mathbb{P}^3$ ramified along a surface $S \subset \mathbb{P}^3$ of degree $2r$. However, we confine our consideration to the case when the surface S has only simple double points, *i.e.*, nodes.

In the case where a 3-fold X with mild singularities is a Fano, a hypersurface of $\mathbb{P}^4$, or a double cover of $\mathbb{P}^3$, the $\mathbb{Q}$-factoriality is equivalent to the global topological property

$$\mathrm{rank}(H^2(X,\mathbb{Z})) = \mathrm{rank}(H_4(X,\mathbb{Z})).$$

In particular, the $\mathbb{Q}$-factoriality implies the factoriality in a hypersurface of $\mathbb{P}^4$ and a double cover of $\mathbb{P}^3$. On the other hand, existence of small resolutions on a variety implies the non-$\mathbb{Q}$-factoriality of the variety.

Let X be a double cover of $\mathbb{P}^3$ ramified along a nodal surface S of degree $2r$. Then, the 2nd integral cohomology of X is isomorphic to the Picard group of X. Therefore, the rank of the group $H^2(X,\mathbb{Z})$ is 1. However, it is not simple to compute the rank of the 4th integral homology group of X. Fortunately, a method to compute the ranks of the 4th integral homology groups of double covers of $\mathbb{P}^3$ ramified along nodal surfaces has been introduced by H. Clemens. The method reduces the topological problem to a rather simple combinatorial problem. To be precise, the ranks can be obtained by studying the number of singular points of S, their position in $\mathbb{P}^3$, and the linear subsystems of $|\mathcal{O}_{\mathbb{P}^3}(3r-4)|$.

Using this method, we can give a partial answer to the following conjecture proposed by I. Cheltsov and the speaker;

Conjecture. *Let $S \subset \mathbb{P}^3$ be a nodal surface of degree $2r$. Suppose that the surface S has at most $r(2r-1)+1$ singular points. Then the double cover of $\mathbb{P}^3$ ramified along S is not $\mathbb{Q}$-factorial if and only if the surface*

S is defined by an equation

$$f_r(x,y,z,w)^2 + h_1(x,y,z,w)g_{2r-1}(x,y,z,w) = 0,$$

where f_r, g_{2r-1}, h_1 are homogeneous polynomials of degree r, $2r-1$, 1, respectively.