Local Cohomology and the Homological Conjectures in Commutative Algebra

During the past few decades a considerable amount of effort has been spent on a set of problems in Commutative Algebra known as the "Homological Conjectures". These questions originated from Serre's homological theory of intersection multiplicities defined in terms of Euler characteristics of Tor functors. Numerous conjectures arose from this work concerning intersection multiplicities as well as related properties of modules of finite projective dimension and Cohen-Macaulay rings and modules. In these talks we describe some of the history of these problems, their current status, and some recent developments in this area. We will concentrate on progress related to local cohomology.

We first state the Monomial Conjecture of Mel Hochster; this conjecture implies several other ones and is at present still not known for rings of mixed characteristic.

Conjecture: Let A be a Noetherian local ring of Krull dimension d, and let $x_1, \ldots, x_d$ be a system of parameters for A (so that the quotient $A/(x_1, \ldots, x_d)$ is an Artinian ring). Then for all integers $t > 0$, we have

$$x_1^t x_2^t \cdots x_d^t \notin (x_1^{t+1}, x_2^{t+1}, \ldots, x_d^{t+1}),$$

where $(x_1^{t+1}, x_2^{t+1}, \ldots, x_d^{t+1})$ denotes the ideal generated by the elements x_i^{t+1}.

This conjecture is rather trivial in characteristic zero, but it is not trivial in positive or mixed characterisitic, and, as mentioned above, has consequences for other questions in those cases.

We recall that a ring is *Cohen-Macaulay* if a system of parameters forms a regular sequence, so that if $x_1 \ldots, x_d$ is a system of parameters, then if $ax_i \in (x_1, \ldots, x_{i-1})$, then $a \in (x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, d$.

If the elements $x_1, \ldots, x_d$ form a regular sequence, then it is a straighforward exercise to show that if $x_1^t x_2^t \cdots x_d^t \in (x_1^{t+1}, x_2^{t+1}, \ldots, x_d^{t+1})$, then $1 \in (x_1, \ldots, x_d)$. Thus if the ring is Cohen-Macaulay, the Monomial Conjecture is clear. By extending these ideas, it is easy to show that the existence of Cohen-Macaulay modules and algebras implies this conjecture.

A related approach to these questions is via local cohomology. If I is the ideal generated by a system of parameters $x_1, \ldots, x_d$, the local cohomology with support in I is the same as the local cohomology with support in the maximal ideal. Thus the local cohomology with support in m is the direct limit of the homology of Koszul complexes on powers of the x_i, and the ring is Cohen-Macaulay if and only if the local cohomology $H_m^i(A)$ vanishes for $i < d$. We also mention that even in the non-Cohen-Macaulay case there are close relations bewteen the annihilators of local cohomolgy and the annihilators of homology of a Koszul complex on a system of parameters.

The most significant recent advance on the Monomial Conjecture was its proof for rings of mixed characteristic of dimension three by Ray Heitmann in 2002. The proof was based on a lemma that said that for a complete Noetherian domain A of dimension 3 of mixed characteristic p, the local cohomology $H_m^2(A^+)$ is annihilated by $p^{1/n}$ for all $n > 0$. Here A^+ denotes the absolute integral closure of A; that is, the integral closure of A in the algebraic closure of its quotient field. The fact that A^+ is integrally closed implies that $H^0_m(A^+)$ and $H^1_m(A^+)$ are zero, so, while this lemma does not imply that A^+ is Cohen-Macaulay, it comes close, and we say that A^+ is "almost Cohen-Macaulay" (we give a more general definition of this term below). It is still an open question whether A^+ is Cohen-Macaulay in this situation. Heitmann proved, however, that it suffices to prove that A^+ is almost Cohen-Macaulay to prove the Monomial Conjecture for the ring A.

We next give a precise definition of "almost Cohen-Macaulay". In the remainder of the abstract, A will denote a Noetherian integrally closed domain of dimension d, and A^+ will denote its absolute integral closure.

Definition: We say that A^+ is *almost Cohen-Macaulay* if there is a valuation v on A^+ which is positive on m such that for every $i < d$, for every $\eta \in H^i_m(A^+)$ for $i < d$, and for every $\epsilon > 0$, there is an element x in A^+ with $v(x) < \epsilon$ and $x\eta = 0$.

The theorem of Heitmann implies that this condition holds for rings of mixed characteristic of dimension 3, where the element x can be taken to be $p^{1/n}$ for sufficiently large n. Also, Heitmann showed that this condition suffices to prove the Monomial Conjecture.

It is an open question whether A^+ is almost Cohen-Macaulay for all complete integrally closed domains A. In these talks we will discuss this question in three cases, depending on the characteristic of A.

1. A has positive characteristic. In this case it is easy to show, using classical techniques with the Frobenius map, that A^+ is almost Cohen-Macaulay. However, Hochster and Huneke have proven the much deeper result that A^+ is actually Cohen-Macaulay in this case.

2. A contains a field of characteristic zero. In this case A^+ is never Cohen-Macaulay if its dimension is at least 3. It is also true that the relevant homological conjectures are known in this case; however, this question appears to be interesting in itself, and it would give a proof of some of these conjectures that does not involve reduction to positive characteristic.

In this case there are some partial results for $H^2_m(A^+)$ done in joint work with Anurag Singh and V. Srinivas. First, for several examples in dimension 3 we were able to show that annihilators of small valuation exist, and these examples seem to indicate that, unlike in the mixed characteristic case, the annihilator x depends strongly on the element η of local cohomology. The most general theorem was found by Srinivas, who showed that for standard graded rings defining a normal scheme, with the valuation v defined by the grading, such elements x can be found by mapping to the Albanese variety and pulling back the map defining multiplication by an integer N. We discuss this method and the facts on abelian varieties that are used.

For more general rings, and for $H^i_m(A^+)$ with $i > 2$, almost nothing is known on this question.

3. A has mixed characteristic. This is the most interesting case, since the Monomial Conjecture is open in this case. In these talks we will say a little about Heitmann's result and describe a new approach to the question using the "Fontaine ring" of a ring of mixed characteristic. This construction has been used in Arithmetic Geometry to reduce questions to the case of positive characteristic, but has not been studied in the context of classical problems in Commutative Algebra. We discuss the questions that arise in attempting to use this method to investigate properties of local cohomology.