ARTIN FUNCTION AND DIOPHANTINE APPROXIMATION IN FIELDS OF POWER SERIES

In 1968, Michael Artin proved the following theorem:

Theorem 0.1. [Ar] *Let I be an ideal of $A[X_1, ..., X_n]$, where A is local, henselian and excellent. Then there exists $\beta : \mathbb{N} \longrightarrow \mathbb{N}$ such that:*
For all $i \in \mathbb{N}$ and for all $(x_1, ..., x_n) \in A^n$ such that

$$f(x) \in \mathfrak{m}^{\beta(i)+1} \textit{ for all } f \in I,$$

there exists $(\overline{x}_1, ..., \overline{x}_n) \in A^n$ such that

$$f(x) = 0 \textit{ for all } f \in I$$

and

$$x_j - \overline{x}_j \in \mathfrak{m}^{i+1} \textit{ for all } j.$$

We will call the **Artin function** of I the lowest function that verifies this theorem. The Artin function of I is an invariant of the A-morphism $A \longrightarrow \frac{A[X]}{I}$. There are three basic examples:

i) If there does not exist $x = (x_1, ..., x_n) \in A^n$ such that $f(x) = 0$ for all $f \in I$, then the Artin function of I constant.
ii) If $A \longrightarrow \frac{A[X]}{I}$ is smooth, then the Artin function of I is equal to the identity.
iii) If I does not verify i) and ii), then the Artin function of I is strictly bigger, as a numerical function, than the identity.

So this function is a "measure" of the smoothness of $A \longrightarrow \frac{A[X]}{I}$.

M. Greenberg [Gr] proved in 1966, in case A is local, henselian, excellent and, moreover, is a discrete valuation ring (at least in characteristic zero), that the Artin function of an ideal I is always bounded by an affine function. M. Spivakovsky [Spi] conjectured at the end of the 80's that this result was true in general. Few cases of this conjecture have been proved by S. Izumi [Iz], and D. Delfino and I. Swanson [DS].

In this talk we will present a new approach to this problem:
Let $\mathcal{O}_N := \Bbbk[[T_1, ..., T_N]]$ where $\Bbbk$ is a field, $\mathfrak{m}$ its maximal ideal and ord the $\mathfrak{m}$-adic order on $\mathcal{O}_N$. We will denote $V_N := \left\{ \frac{x}{y} \ / \ x, y \in \mathcal{O}_N, \ \mathrm{ord}(x) \geq \mathrm{ord}(y) \right\}$ the discrete valaution ring which dominates $\mathcal{O}_N$ for ord, and we will denote $\widehat{V}_N := \Bbbk\left(\frac{T_1}{T_N}, ..., \frac{T_{N-1}}{T_N}\right)[[T_N]]$ its completion for the Krull topology. Let $\mathbb{K}_N$

and $\widehat{\mathbb{K}}_N$ be the fraction fields of $\mathcal{O}_N$ and $\widehat{V}_N$ respectively. The valuation ord extends to $\mathbb{K}_N$ and $\widehat{\mathbb{K}}_N$. This valuation defines a norm, denoted by $|\,|$: $\forall x \in \widehat{\mathbb{K}}_N$, $|x| := e^{-ord(x)}$. We can prove the following theorem of diophantine approximation:

Theorem 0.2. [Ro3] *Let $z \in \widehat{\mathbb{K}}_N \backslash \mathbb{K}_N$ algebraic over $\mathbb{K}_N$. Then there exist K and a such that*

$$\forall x,\, y \in \mathcal{O}_N,\ \left| z - \frac{x}{y} \right| \geq K|y|^a.$$

We will see that this theorem is equivalent to the next one :

Theorem 0.3. [Ro3] *Let $P(X,\, Y)$ an homogeneous polynomial of $\mathcal{O}_N[X,\, Y]$ such that $(0,\, 0)$ is the only one zero of P in $\mathcal{O}_N$. Then the Artin function of P is bounded by an affine function.*

We will see that the constant a of theorem 0.2 cannot be choosen equal to the degree of the extension $\mathbb{K}_N \longrightarrow \mathbb{K}_N[z]$, as for the classical theorem of diophantine approximation of Liouville. We will deduce of this a counter-example to the conjecture of M. Spivakovsky:

Theorem 0.4. [Ro1] *The Artin function of $X^2 - ZY^2 \in \mathcal{O}_N[X,\, Y,\, Z]$ is bounded from below by a polynomial equation of degree two.*

References

[Ar] M. Artin, Algebraic approximation of structures over complete local rings, *Publ. Math. IHES*, **36**, (1969), 23-58.

[DS] D. Delfino - I. Swanson, Integral closure of ideals in excellent local rings, *J. Algebra*, **187**, (1997), 422-445.

[Gr] M. J. Greenberg, Rational points in henselian discrete valuation rings, *Publ. Math. IHES*, **31**, (1966), 59-64.

[Hi] M. Hickel, Fonction de Artin et germes de courbes tracées sur un germe d'espace analytique, *Am. J. of Math.*, **115**, (1993), 1299-1334.

[Iz] S. Izumi, A mesure of integrity for local analytic algebras, *Publ. RIMS, Kyoto Univ.*, **21**, (1985), 719-736.

[PP] G. Pfister - D. Popescu, Die strenge Approximationeigenschaft lokaler Ringe, *Invent. Math.*, **30**, (1975), 145-174.

[Ro1] G. Rond, Contre-exemple à la linéarité de la fonction de Artin, preprint ArXiv, (2004).

[Ro2] G. Rond, Lemme d'Artin-Rees, théorème d'Izumi et fonctions de Artin, *J. Algebra*, to appear.

[Ro3] G. Rond, Approximation diophantienne dans les corps de series en plusieurs variables, preprint ArXiv, (2005).

[Spi] M. Spivakovsky, Valuations, the linear Artin approximation theorem and convergence of formal functions, Proceedings of the II SBWAG, Santiago de Compostela, Felipe Gago and Emilio Villanueva, editors, ALXEBRA, **54**, (1990), 237-254.