

Complete intersection points on affine varieties and zero cycles

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Let $X = \text{Spec } A$ be an irreducible affine variety of dimension d over an algebraically closed field k , so that the coordinate ring A of algebraic functions on X is a finitely generated k -algebra which is an integral domain of Krull dimension d . A *complete intersection point* of X is a point $x \in X$ such that the maximal ideal $\mathfrak{M}_x \subset A$ of functions vanishing at x is generated by d elements $f_1, \dots, f_d \in \mathfrak{M}_x$. Geometrically, this means that $x \in X$ is a non-singular point, and the hypersurfaces $H_i = \{y \in X \mid f_i(y) = 0\}$ satisfy $H_1 \cap \dots \cap H_d = \{x\}$, and the H_i intersect transversally near x .

More generally, if $X = \text{Spec } A$ is a reduced affine k -variety of dimension d , a point x is a complete intersection point if \mathfrak{M}_x has height d , and is generated by d elements. Define a point $x \in X$ to be a non-singular point if its local ring $\mathcal{O}_{X,x}$ is a regular local ring of dimension d ; clearly a complete intersection point is non-singular.

We are interested here in characterizing varieties $X = \text{Spec } A$ such that all non-singular points $x \in X$ are complete intersections. This problem turns out to have different flavours, depending on the ground field k , and is related to interesting conjectures in the theory of algebraic cycles, and thereby to algebraic K-theory.

In this talk, I will give an introduction to this topic, dwelling in particular on some recent results based on the paper [3], and some applications to complete intersections in [4].

Here are two results, proved in [4], using the results of [3]. Let $\overline{\mathbb{F}}_p$ denote the algebraic closure of a finite field of char. p , and let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers (algebraic closure of \mathbb{Q} in \mathbb{C}).

Theorem 1. *Let A be a finitely generated reduced $\overline{\mathbb{F}}_p$ -algebra of dimension $d \geq 2$. Then any non-singular point of $X = \text{Spec } A$ is a complete intersection point.*

Theorem 2. *Let $A = \bigoplus_{n \geq 0} A_n$ be a reduced, finitely generated $\overline{\mathbb{Q}}$ -algebra of dimension $d \geq 2$. Then any non-singular point of $X = \text{Spec } A$ is a complete intersection point.*

This leads to the following interesting examples (of ‘‘Bloch-Beilinson type’’). Let $A = \mathbb{C}[x, y, z]/(f(x, y, z))$ be the homogeneous coordinate ring of a nonsingular plane curve of degree $d \geq 4$, where f has coefficients in $\overline{\mathbb{Q}}$. A non-singular point $(a, b, c) \in X = \text{Spec } A \subset \mathbb{A}_{\mathbb{C}}^3$ is a complete intersection if and only if the point $[a : b : c] \in \text{Proj } A \subset \mathbb{P}_{\mathbb{C}}^2$ is a $\overline{\mathbb{Q}}$ -rational point (i.e., (a, b, c) is proportional to a triple of algebraic numbers).

Some survey articles giving background, detailed references, and explaining further the connections with commutative algebra, are [2] and [1].

References

- [1] V. Srinivas, *Zero cycles on singular varieties*, in *The Arithmetic and Geometry of Algebraic Cycles*, ed. B. B. Gordon *et al.*, NATO Science Series Vol. C 548, Kluwer (2000), pp. 347-382.
- [2] V. Srinivas, *Some Geometric Methods in Commutative Algebra*, in *Computational Commutative Algebra and Combinatorics (Osaka, 1999)*, Advanced Studies in Pure Math. 33 (2002) 231-276.
- [3] Amalendu Krishna, V. Srinivas, *Zero cycles and K-theory on normal surfaces*, Annals of Math. 156 (2002) 155-195.
- [4] Amalendu Krishna, V. Srinivas, *Zero cycles on singular varieties*, to appear in Proceedings of EAGER Conference *Workshop Algebraic Cycles and Motives*, Leiden, 2004.