Lefschetz Theorems for the divisor class group

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Let k be an algebraically closed field, and let $A = \oplus_{n\geq 0}A_n$ be a standard graded k-algebra (i.e. $A_0 = k$, and A is generated over k by A_1, which is a finite dimensional k-vector space). When A is a domain, of Krull dimension $d \geq 2$, which is regular in codimension 1, we let $\mathrm{Cl}\,(A)$ denote the *divisor class group* of A, which coincides with the Chow group $CH_{d-1}(A)$ ([F]).

If now $\dim A \geq 3$, and $h \in A_1$ is any non-zero element, such that A/hA is again regular in codimension 1. let $B = (A/hA)_{\mathrm{red}}$. There is an induced "restriction homomorphism" $\mathrm{Cl}\,(A) \to \mathrm{Cl}\,(B)$, which may be interpreted as a special case of Fulton's *refined Gysin homomorphism* on Chow groups, defined in [F]. The general "Lefschetz problem" I would like to discuss here is the following: under what conditions can we say that $\mathrm{Cl}\,(A) \to \mathrm{Cl}\,(B)$ is an *isomorphism*?

In geometric language, let $X = \mathrm{Proj}\,A$, $Y = \mathrm{Proj}\,B$, so that Y is an irreducible hyperplane section of the projective variety X in a suitable projective embedding. There is an induced homomorphism (see [Ha] II for example) $\mathrm{Cl}\,(X) \to \mathrm{Cl}\,(Y)$, which may also be interpreted as a refined Gysin map of Fulton [F]. There are also isomorphisms $\mathrm{Cl}\,(A) = \mathrm{Cl}\,(X)/\mathbb{Z}$, $\quad \mathrm{Cl}\,(B) = \mathrm{Cl}\,(Y)/\mathbb{Z}$, compatible with the respective restriction homomorphisms, where the subgroup $\mathbb{Z}$ in each case is the cyclic subgroup generated by class of a hyperplane section. Thus, the Lefschetz problem for the pair of rings (A, B) is equivalent to the "geometric Lefschetz problem" for $\mathrm{Cl}\,(X) \to \mathrm{Cl}\,(Y)$.

Now suppose X, Y are non-singular projective varieties over $\mathbb{C}$, the complex numbers. By transcendental arguments (the *Lefschetz hyperplane theorem*, combined with Serre's GAGA, Hodge theory etc.), one has that if $\dim A \geq 5$, or equivalently, $\dim X \geq 4$, then $\mathrm{Cl}\,(A) \to \mathrm{Cl}\,(B)$ is an isomorphism, and $\mathrm{Cl}\,(X) \to \mathrm{Cl}\,(Y)$ is an isomorphism.

When $\dim A = 4$, $k = \mathbb{C}$, and X, Y are non-singular, the Lefschetz hyperplane theorem similarly implies that $\mathrm{Cl}\,(X) \to \mathrm{Cl}\,(Y)$, and hence also $\mathrm{Cl}\,(A) \to \mathrm{Cl}\,(B)$, are *injective*, with finitely generated cokernel. There are simple examples to show that this need not be an isomorphism.

Assume further that the projective embedding of X is by a "sufficiently ample" line bundle; more concretely, assume that

$$\dim_k H^2(X, \mathcal{O}_X) < \dim_k H^2(Y, \mathcal{O}_Y).$$

This condition is in fact independent of Y, for a Zariski open subset of $h \in A_1$. Again, if $k = \mathbb{C}$, then the *Noether-Lefschetz theorem* implies that, for $h \in A_1 \cong \mathbb{C}^N$ lying outside a countable union of analytic subvarieties, the map $\mathrm{Cl}\,(X) \to \mathrm{Cl}\,(Y)$ is an isomorphism, and $\mathrm{Cl}\,(A) \to \mathrm{Cl}\,(B)$ is an isomorphism as well. In particular, if $X = \mathbb{P}^3_{\mathbb{C}}$, one gets "Noether's theorem", that for a "general" hypersurface $Y \subset X$ of degree $d \geq 4$, we have $\mathbb{Z} = \mathrm{Cl}\,(X) \cong \mathrm{Cl}\,(Y)$.

This is equivalent to the statement that the homogeneous coordinate ring of such a hypersurface is a UFD. The "classical" proof of the Noether-Lefschetz theorem uses, in addition to GAGA and Hodge theory, the monodromy theory of Lefschetz pencils (vanishing cycles, Picard-Lefschetz formula, etc.)

From an algebraic standpoint, (i) we'd like to give purely algebraic formulations and proofs of the above theorems, replacing $\mathbb{C}$ by an arbitrary algebraically closed field k if possible (ii) we'd like to eliminate the restrictive hypothesis that $X = \operatorname{Proj} A$ is non-singular.

The case when $\dim A \geq 5$, but X is still assumed non-singular, and k has characteristic 0, is due to Grothendieck [SGA2]. The *Grothendieck-Lefschetz theorem* states that, if $k = \bar{k}$ is of characteristic 0, $\dim A \geq 5$, and $X = \operatorname{Proj} A$ is nonsingular, then for a Zariski open set of $h \in A_1$, the restriction map $\operatorname{Cl}(A) \to \operatorname{Cl}(B)$ is an isomorphism. In characteristic $p > 0$, the method yields that the kernel and cokernel of the above map are p-power torsion groups.

The Noether-Lefschetz theorem has the following purely algebraic formulation. For any extension field L of k, let $A_L = A \otimes_k L$, which is a standard graded L-algebra. Let V be the dual vector space to A_1, and let K be the algebraic closure of the quotient field of the symmetric algebra of V. There is a "generic" linear form $h \in (A_K)_1 = A_1 \otimes_k K$, corresponding to the identity element in $A_1 \otimes_k V = \operatorname{End}_k(A_1)$. Let $B = A_K/hA_K$. Then, under appropriate "ampleness" hypothesis, we want that $\operatorname{Cl}(A_K) \to \operatorname{Cl}(B_K)$ is an isomorphism. There are "algebraic" results of this type, proved in [SGA7-II]. Independently, and earlier, results of A. Weil (see [W]) imply that in any characteristic, with the above notation, $\operatorname{Cl}(A_K) \to \operatorname{Cl}(B_K)$ is always injective, with finitely generated cokernel, if $\dim A \geq 4$.

Now I'd like to come to some new results, which are joint work with G. V. Ravindra. At present, these are all in characteristic 0, though there are hopes to obtain some results in characteristic p as well, in due course. Henceforth, k will denote an algebraically closed field of characteristic 0.

Theorem 1. *With the above notation, for a Zariski open set of $h \in A_1$, the restriction map*

$$\operatorname{Cl}(A) \to \operatorname{Cl}(B)$$

is an isomorphism, if $\dim A \geq 5$*, and is injective, with finitely generated cokernel, if* $\dim A = 4$*. Equivalently,* $\operatorname{Cl}(X) \to \operatorname{Cl}(Y)$ *is an isomorphism, if* $\dim X \geq 4$*, and is injective with finitely generated cokernel, if* $\dim X = 3$.

The proof is similar to that of the Grothendieck-Lefschetz theorem in [Ha2], using the theory of formal schemes, and Grothendieck's *Lefschetz conditions.* The treatment is purely algebraic, but uses resolution of singularities, as well as cohomology vanishing theorems which hold only in characteristic 0. Thus, as it stands, one cannot get from our proof the statement of isomorphism upto p-primary torsion, in characteristic p.

When $k = \mathbb{C}$, a different, transcendental proof of Theorem 1 has been given by N. Fakhruddin, using Goresky and MacPherson's stratified Morse theory, and mixed Hodge structures (see the appendix in [RS]). In another direction, Theorem 1 yields an algebraic proof that the Gysin map for Albanese 1-motives (see [1]) is an isomorphism, in appropriate situations; this is in accordance with Deligne's philosophy of 1-motives.

A second result, obtained with G. V. Ravindra, generalizes an unpublished work with Mohan Kumar, for the case $X = \mathbb{P}^3_{\mathbb{C}}$. For a normal projective k-variety X, let K_X denote the coherent sheaf obtained as the direct image of the canonical sheaf of any resolution of singularities (recall k has characteristic 0); this is independent of the resolution, and for $k = \mathbb{C}$ is the sheaf of *locally square integrable forms*, considered by Grauert and Riemenschneider.

Theorem 2. *Let X be a normal projective 3-fold over k, with an ample invertible sheaf $\mathcal{O}_X(1)$, and a base-point free linear system $V \subset H^0(X, \mathcal{O}_X(1))$, giving a morphism $f : X \to \mathbb{P}^N_k$. Assume further that the coherent sheaf $(f_*K_X)(1)$ is generated by its global sections.*

Let K be the algebraic closure of the function field of $\hat{\mathbb{P}}^N_k$ (the dual projective space), and Y_K the geometric generic member of the linear system $|V|$.

Then the restriction map $\mathrm{Cl}\,(X_K) \to \mathrm{Cl}\,(Y_K)$ is an isomorphism.

This implies the Noether theorem for surfaces in $\mathbb{P}^3_{\mathbb{C}}$. but does not give the full strength of the Noether-Lefschetz theorem over $\mathbb{C}$ proved by transcendental arguments. On the other hand, the transcendental arguments, in present form, do not seem to give Theorem 2 in full generality either, since the corresponding monodromy representation is (presumably) rather complicated.

References

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