F-thresholds and Bernstein-Sato polynomials

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This is a joint work with M. Mustaţă and K.-i. Watanabe.

We introduce and study invariants of singularities in positive characteristic called F-thresholds. They give an analogue of the jumping coefficients of multiplier ideals in characteristic zero. Unlike these, however, the F-thresholds are not defined via resolution of singularities, but via the action of the Frobenius map.

We will restrict ourselves to the case of an ambient nonsingular variety, when our invariants have a down-to-earth description. Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. We want to measure the singularities of a nonzero ideal $\mathfrak{a} \subseteq \mathfrak{m}$. For every ideal $J \subseteq \mathfrak{m}$ containing $\mathfrak{a}$ in its radical, and for every $e \geq 1$, we put

$$\nu_{\mathfrak{a}}^J(p^e) := \max\{r | \mathfrak{a}^r \not\subseteq J^{[p^e]}\},$$

where $J^{[p^e]} = (f^{p^e} | f \in J) \subset R$. One can check that the limit

$$\mathrm{c}^J(\mathfrak{a}) := \lim_{e \to \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$$

exists and is finite. We call this limit the F-threshold of $\mathfrak{a}$ with respect to J. When $J = \mathfrak{m}$, we simply write $\mathrm{c}(\mathfrak{a})$ and $\nu_{\mathfrak{a}}(p^e)$. The invariant $\mathrm{c}(\mathfrak{a})$ was introduced in [TW] under the name of F-pure threshold.

We point out that it is not known whether the analogue of two basic properties of jumping coefficients of multiplier ideals hold in our setting: whether $\mathrm{c}^J(\mathfrak{a})$ is always a rational number and whether the set of all F-thresholds of $\mathfrak{a}$ is discrete. Recently Hara proved that this analogue holds true when R is a two-dimensional formal power series ring over a finite field and $\mathfrak{a}$ is a principle ideal (see [H]).

For simplicity, we assume that $\mathfrak{a}$ and J are ideals in $\mathbb{Z}[X_1, \ldots, X_n]$, contained in $(X_1, \ldots, X_n)$ and such that $\mathfrak{a}$ is contained in the radical of J. Let us denote by $\mathfrak{a}_p$ and J_p the localizations at $(X_1, \ldots, X_n)$ of the images of $\mathfrak{a}$ and J, respectively, in $\mathbb{F}_p[X_1, \ldots, X_n]$. We want to compare our invariants mod p (which we write as $\nu_{\mathfrak{a}}^J(p^e)$ and $\mathrm{c}^J(\mathfrak{a}_p)$) with the characteristic zero invariants of $\mathfrak{a}$ (more precisely, with the invariants around the origin of the image $\mathfrak{a}_{\mathbb{Q}}$ of $\mathfrak{a}$ in $\mathbb{Q}[X_1, \ldots, X_n]$).

First, let us denote by $\mathrm{lc}_0(\mathfrak{a})$ the log canonical threshold (which is the smallest jumping coefficient in characteristic zero) of $\mathfrak{a}_{\mathbb{Q}}$ around the origin. It follows from results of Hara and Watanabe (see [HW]) that if $p \gg 0$ then $\mathrm{c}(\mathfrak{a}_p) \leq \mathrm{lc}_0(\mathfrak{a})$ and $\lim_{p\to\infty} \mathrm{c}(\mathfrak{a}_p) = \mathrm{lc}_0(\mathfrak{a})$. Moreover, results of Hara and Yoshida from [HY] allow the extension of these formulas to higher jumping numbers. It is easy to give examples in which $\mathrm{c}(\mathfrak{a}_p) \neq \mathrm{lc}_0(\mathfrak{a})$ for infinitely many p. On the other hand, one conjectures that there are infinitely many p with $\mathrm{c}(\mathfrak{a}_p) = \mathrm{lc}_0(\mathfrak{a})$.

A surprising fact is that our invariants for $\mathfrak{a}_p$ are related to the Bernstein-Sato polynomial $b_{\mathfrak{a},0}(s)$ of $\mathfrak{a}$. More precisely, we show that for all $p \gg 0$ and for all e, we have $b_{\mathfrak{a},0}(\nu_{\mathfrak{a}}^J(p^e)) \equiv 0 \pmod{p}$. We show on some examples how to use this to give roots of the Bernstein-Sato polynomial (and not just roots mod p). In these examples we will see the following behavior: given some ideal J containing $\mathfrak{a}$ in its radical, and $e \geq 1$, we can find N such that for all $i \in \{1, \ldots, N-1\}$ relatively prime to N there are polynomials $P_i \in \mathbb{Q}[t]$ of degree e satisfying $\nu_{\mathfrak{a}}^J(p^e) = P_i(p)$ for all $p \gg 0$, with $p \equiv i \pmod{N}$. The previous observation implies that $b_{\mathfrak{a},0}(P_i(0))$ is divisible by p for every such p. By Dirichlet's Theorem we deduce that $P_i(0)$ is a root of $b_{\mathfrak{a},0}$.

An interesting question is which roots can be obtained by the above method. It is shown in [BMS] that for monomial ideals the functions $p \to \nu_{\mathfrak{a}}^J(p^e)$ behave as described above, and moreover, all roots of the Bernstein-Sato polynomial are given by this procedure. On the other hand, in general there are roots which cannot be given by our method.

References

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