HILBERT-KUNZ MULTIPLICITIES AND VECTOR BUNDLES OVER CURVES

V. TRIVEDI

1. ABSTRACT

Let $(R, \mathbf{m})$ be a Noetherian local ring (respectively a standard graded ring over a field k with the graded maximal ideal $\mathbf{m}$) of dimension d and of prime characteristic $p > 0$, and let I be an $\mathbf{m}$-primary ideal (respectively a homogeneous ideal of R, of finite colength). Then one defines the *Hilbert-Kunz function* of R with respect to I as

$$HK_{R,I}(p^n) = \ell(R/I^{(p^n)}),$$

where

$$I^{(p^n)} = n\text{-th Frobenius power of } I$$
$$= \text{ideal generated by } p^n\text{-th powers of elements of } I.$$

The associated *Hilbert-Kunz multiplicity* (HK multiplicity for the sake of abberviation) is defined to be

$$e_{HK}(R, I) = \lim_{n\to\infty} \frac{HK_{R,I}(p^n)}{p^{nd}}.$$

In the graded situation we denote alternatively HK multiplicity $e_{HK}(R, I)$ by $e_{HK}(X, \mathcal{L}, I)$, where $X = \text{Proj } R$ and $\mathcal{L} = \mathcal{O}_X(1)$ is the very ample line bundle on X and I is a homogeneous ideal in the homogeneous coordinate ring of X, of finite colength.

In case I is the maximal ideal of R we denote HK multiplicity by $e_{HK}(R)$ or $e_{HK}(X, \mathcal{L})$.

In this talk we discuss HK multiplicity of two dimensional graded ring R (of projective curve) over a field k of char $p > 0$, with respect to a homogeneous ideal I of finite colength. Without loss of generality one can assume that the ring is a domain and k is algebraically closed. Let I be generated by homogeneous generators $f_1, \ldots, f_k$ of degrees $d_1, \ldots, d_k$ respectively. Consider the following short exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \longrightarrow V_{\mathcal{L},I} \longrightarrow \oplus_i \mathcal{O}_X(1-d_i) \longrightarrow \mathcal{O}_X(1) \mapsto 0,$$

where $\mathcal{O}_X(1-d_i) \to \mathcal{O}_X(1)$ is given by multiplication by f_i. Then $\pi^*(V_{\mathcal{L},I})$ is a vector bundle of rank $k-1$ on $\widetilde{X}$, where $\pi : \widetilde{X} \to X$ is the normalization of X.

We express HK multiplicity in terms of (i) "standard" invariants of the curve which are constant in a flat family and (ii) normalized slopes of the quotients occuring in a strongly semistable Harder-Narasimhan filtration (HN filtration) of $\pi^*(V_{\mathcal{L},I})$ on $\widetilde{X}$. As a consequence we get

Theorem 1.1. (1) *$e_{HK}(R, I)$ is a rational number.*

(2) *Moreover if $f_1, \ldots, f_k$ are minimal generators of I then*

$$e_{HK} \geq d/2((\sum_i d_i)^2/(t-1) - \sum_i d_i^2)$$

and the equality holds if and only if $V = \pi^* V_{X,\mathcal{L}}$ is strongly semistable, i.e., $F^{s*}(V)$ is semistable, for all $s \geq 0$, on $\widetilde{X}$, where $F^s : \widetilde{X} \to \widetilde{X}$ is the s-fold iterated Frobenius map.

1991 *Mathematics Subject Classification.* 13D40.

(The first part of the above theorem was also proved independently by Brenner, for general d_i and by the speaker for $d_i = 1$, giving essentially the same proof).

In case of plane curves, when $p > d(d-3)$, we have a simple numerical characterization of semistablity of the Frobenius pullback of the kernel bundle $V_{\mathcal{L},I}$, via HK multiplicity. In particular we prove the following.

Theorem 1.2. *Let X be an irreducible plane curve of degree d, i.e., $X = \text{Proj } R = k[x,y,z]/(f)$, where f is an irreducible homogeneous polynomial of degree d. Then, there exist integers l and s such that one of the following occurs:*

(1) $e_{HK}(X, \mathcal{O}_X(1)) = \frac{3d}{4}$
(2) $e_{HK}(X, \mathcal{O}_X(1)) = \frac{3d}{4} + \frac{l^2}{4dp^{2s}}$, *where $s \geq 1$ and $l \equiv pd(\text{mod } 2)$ with $0 < l \leq d(d-3)$.*
(3) $e_{HK}(X, \mathcal{O}_X(1)) = \frac{3d}{4} + \frac{l^2}{4d}$, *where $l \equiv d(\text{mod } 2)$ and $0 < l \leq d$.*
(4) *Moreover, if X is nonsingular then the only possible HK multiplicities are of type (1) or (2).*
(5) *If X has a singular point of multiplicity $r \geq d/2$ then*

$$e_{HK}(X, \mathcal{O}_X(1)) = \frac{3d}{4} + \frac{(2r-d)^2}{4d}.$$

Examples computed so far show that $e_{HK}(R, I)$ can depend on char p of the ring, and a natural question is to ask how the HK multiplicities of reductions (mod p) of a given variety (in char. 0) vary with p. To make this more precise consider the following situation. Let k be an algebraically closed field of characteristic 0. Let R be a finitely generated $\mathbb{N}$-graded two dimensional domain over k. Let $I \subset R$ be an homogeneous ideal of finite colength. For the pair (R, I) we take a spread (A, R_A, I_A), which means we choose a finitely generated $\mathbb{Z}$-algebra $A \subseteq k$, a finitely generated $\mathbb{N}$-graded algebra R_A over A and an homogeneous ideal $I_A \subset R_A$ such that $R_A \otimes_A k = R$ and for any closed point $s \in \text{Spec } A$ (*i.e.* maximal ideal of A) the ring $R_s = R_A \otimes_A k(s)$ is a finitely generated $\mathbb{N}$-graded 2-dimensional domain (which is a normal domain if R is normal) over $k(s)$ and the ideal $I_s = \text{Im}(I_A \otimes_A k(s)) \subset R_s$ is an homogeneous ideal of finite colength.

We prove the following result.

Theorem 1.3. *Let R be a standard graded two dimensional domain over k. Let $I \subset R$ be an homogeneous ideal of finite colength. Let (A, R_A, I_A) be a spread as given above. Then*

$$\lim_{s \to s_0} e_{HK}(R_s, I_s)$$

exists, where $s_0 = \text{Spec } Q(A)$ is the generic point of Spec A, *and the limit is taken over closed points $s \in \text{Spec}A$.*

The proof is again reduction to a problem on arbitrary vector-bundles over nonsingular projective curves. We prove the following key lemma (which is not true for arbitrary p), making crucial use of a result by Shepherd-Barron.

Lemma 1.4. *If $p \geq 4(genus(X) - 1)(\text{rank } V)^3$, then the HN filtration of F^*V is a refinement of the Frobenius pull back of the HN filtration of V,*

We analyse the HN (Harder-Narasimahan) polygons $HNP_{p^s}(V)$ which corresponds to the iterated Frobenius pull back $F^{s*}V$ of vector bundle V on X. Using the above result we deduce that as $p \to \infty$ the polygon

$$HNP_{p^s}(V) \to HNP(V).$$

In particular the Hilbert-Kunz multiplicities of the reductions to positive characteristics of an irreducible projective curve in characteristic 0 have a well-defined limit as the characteristic tends to ∞. This limit, which is (relatively) an easier invariant to compute, is a *lower bound* for the HK multiplicities of the reductions (mod p), though examples of Monsky show that the convergence is not monotonic as $p \to \infty$, in general.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai-400005, India

E-mail address: vija@math.tifr.res.in