Affine braid group actions on derived categories of algebraic surfaces

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1 Autoequivalences of derived categories

Let X be a smooth projective variety over $\mathbb{C}$. The derived category $D(X)$ of X is a triangulated category whose objects are bounded complexes of coherent sheaves on X. By an equivalence $D(Y) \simeq D(X)$, we always mean a $\mathbb{C}$-linear equivalence of triangulated categories.

We denote the group of isomorphism classes of autoequivalences of $D(X)$ by $\operatorname{Auteq} D(X)$.

$$\operatorname{Auteq} D(X) := \left\{ \Phi : D(X) \simeq D(X) \text{ autoequivalence } \right\} / \cong$$

It seems natural to consider the following problem.

Problem 1.1. Describe $\operatorname{Auteq} D(X)$ for smooth projective varieties X.

We note that $\operatorname{Auteq} D(X)$ always contains the group

$$A(X) := (\operatorname{Aut} X \ltimes \operatorname{Pic} X) \times \mathbb{Z},$$

generated by functors of tensoring with invertible sheaves, automorphisms of X and the shift functor. When K_X or $-K_X$ is ample, it is shown that $\operatorname{Auteq} D(X) \cong A(X)$ in [BO01]. When X is an abelian variety, Orlov solves Problem 1.1 in [Or02]. In this case, $\operatorname{Auteq} D(X)$ is strictly larger than $A(X)$.

1.1 Spherical objects and twist functors

The *twist functors* along *spherical objects* are autoequivalences of another kind that are not in $A(X)$. Let us recall the definition of them.

For an object $\mathcal{P} \in D(X \times Y)$, an *integral functor*

$$\Phi^{\mathcal{P}}_{X \to Y} : D(X) \to D(Y)$$

is defined by

$$\Phi^{\mathcal{P}}_{X \to Y}(-) = \mathbf{R}\pi_{Y*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi_X^*(-)),$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the projections.

Definition 1.2 ([ST01]). (i) We say that an object $\alpha \in D(X)$ is *spherical* if we have $\alpha \otimes \omega_X \cong \alpha$ and

$$\operatorname{Hom}^k_{D(X)}(\alpha, \alpha) \cong \begin{cases} 0 & k \neq 0, \dim X \\ \mathbb{C} & k = 0, \dim X. \end{cases}$$

(ii) Let $\alpha \in D(X)$ be a spherical object. We consider the mapping cone

$$\mathcal{C} = Cone(\pi_1^*\alpha^\vee \overset{\mathbf{L}}{\otimes} \pi_2^*\alpha \to \mathcal{O}_\Delta)$$

of the natural evaluation $\pi_1^*\alpha^\vee \overset{\mathbf{L}}{\otimes} \pi_2^*\alpha \to \mathcal{O}_\Delta$, where $\Delta \subset X \times X$ is the diagonal, and π_i is the projection of $X \times X$ to the i-th factor. Then the integral functor $T_\alpha := \Phi^{\mathcal{C}}_{X\to X}$ defines an autoequivalence of $D(X)$, called the *twist functor* along the spherical object α.

By the definition of twist functors T_α, for any object $\beta \in D(X)$, we have a triangle

$$\mathbf{R}\operatorname{Hom}_{D(X)}(\alpha, \beta) \overset{\mathbf{L}}{\otimes}_{\mathbb{C}} \alpha \longrightarrow \beta \longrightarrow T_\alpha(\beta)$$

so that, at the level of K-theory,

$$[T_\alpha(\beta)] = [\beta] - \chi(\alpha, \beta)[\alpha].$$

Example 1.3. (i) Let X be a K3 surface and $\mathcal{L}$ a line bundle on Z. Then $\mathcal{L}$ is a spherical object of $D(X)$.

(ii) Let Z be the fundamental cycle of -2-curves in ADE configurations on a smooth surface X and $\mathcal{L}$ a line bundle on Z. Then $\mathcal{L}$ is a spherical object of $D_Z(X)$. Here $D_Z(X)$ denotes the full subcategory of $D(X)$ consisting of objects supported on Z.

1.2 Results in [IU04].

In our paper [IU04], we consider a chain Z of -2-curves on a smooth surface X and study the autoequivalences of the derived category $D_Z(X)$.

Note that the twist functor T_α can be defined as long as the support of α is projective, even if X is not projective. Moreover, the category $D_Z(X)$ depends only on the formal neighborhood of Z in X. Thus we can assume as follows:

$$Y = \operatorname{Spec} \mathbb{C}[[x, y, z]]/(x^2 + y^2 + z^{n+1})$$

is the A_n-singularity,

$$f : X \to Y$$

its minimal resolution and

$$Z = f^{-1}(P) = C_1 \cup \cdots \cup C_n$$

where $P \in Y$ is the closed point.

For an autoequivalence $\Phi \in \operatorname{Auteq} D_Z(X)$, we don't know if it is always isomorphic to an integral functor. Here, an integral functor from $D_Z(X)$ to $D_Z(X)$ is defined by an object $\mathcal{P} \in D(X \times X)$ whose support is projective over X with respect to each projection. If an autoequivalence is given as an integral functor, we call it a *Fourier-Mukai transform* (*FM transform*). Let

$$\operatorname{Auteq}^{\mathrm{FM}} D_Z(X) \subset \operatorname{Auteq} D_Z(X)$$

be the subgroup consisting of FM transforms.

We denote the dualizing sheaf on Z by ω_Z and put

$$B := \left\langle T_{\mathcal{O}_{C_l}(-1)}, T_{\omega_Z} \;\middle|\; 1 \le l \le n \right\rangle \subset \operatorname{Auteq}^{\mathrm{FM}} D_Z(X).$$

Theorem 1.4 ([IU04]). *We have*

$$\operatorname{Auteq}^{FM} D_Z(X) = (\langle B, \operatorname{Pic} X\rangle \rtimes \operatorname{Aut} X) \times \mathbb{Z}.$$

Here $\mathbb{Z}$ *is the group generated by the shift* [1].

Remark 1.5. We know more about subgroups of $\operatorname{Auteq}^{\mathrm{FM}} D_Z(X)$, that is, we have the following:

- $B \cap \operatorname{Pic} X = \langle \otimes\mathcal{O}_X(C_1), \dots, \otimes\mathcal{O}_X(C_n)\rangle$.
- $\langle B, \operatorname{Pic} X\rangle \cong B \rtimes \mathbb{Z}/(n+1)\mathbb{Z}$.
- $B = \left\langle T_\alpha \;\middle|\; \alpha \in D_Z(X),\ \textit{spherical} \right\rangle$.

According to Orlov's theorem [Or97], any autoequivalence $\Phi \in \operatorname{Auteq} D(S)$ for a smooth *projective* variety S is isomorphic to an integral functor $\Phi^{\mathcal{P}}_{S\to S}$ for some $\mathcal{P} \in D(S \times S)$. Using the Orlov's theorem, we obtain Theorem 1.6 below. The proof is similar to the one of Theorem 1.4.

Theorem 1.6 ([IU04]). *Let* S *be a smooth projective surface of general type whose canonical model has* A_n*-singularities at worst. Then we have*

$$\operatorname{Auteq} D(S) = \left\langle T_{\mathcal{O}_C(a)}, A(S) \;\middle|\; C : -2\text{-}\textit{curve},\ a \in \mathbb{Z} \right\rangle.$$

2 Affine braid group actions and Bridgeland's stability conditions

2.1 Affine braid group actions on derived categories

Let us consider the situation in Theorem 1.4. Put $\alpha_i := \mathcal{O}_{C_i}(-1)$ $(1 \le i \le n)$ and $\alpha_0 := \alpha_{n+1} := \omega_Z$. Then B is generated by all T_{α_i}'s by the definition. The result in [ST01] implies that T_{α_i}'s satisfy the following relations.

$$\begin{cases} T_{\alpha_i}T_{\alpha_{i+1}}T_{\alpha_i} \cong T_{\alpha_{i+1}}T_{\alpha_i}T_{\alpha_{i+1}} & \textit{if } 0 \le i \le n, \\ T_{\alpha_i}T_{\alpha_j} \cong T_{\alpha_j}T_{\alpha_i} & \textit{if } i-j \ne \pm 1. \end{cases}$$

In other words, the generators of B satisfy the defining relations of the affine braid group of type $\tilde{A}_n$, which is denoted by $\mathcal{A}(\tilde{A}_n)$. Therefore it follows from Theorem 1.4 that the group $\mathcal{A}(\tilde{A}_n)$ acts on $D_Z(X)$.

Conjecture 2.1 ([IU04]). *The action of the affine braid group* $\mathcal{A}(\tilde{A}_n)$ *on* $D_Z(X)$ *is faithful. Consequently, we have an isomorphism* $B \cong \mathcal{A}(\tilde{A}_n)$.

Actually we can show this conjecture is true when $n = 1$, namely when X is the minimal resolution of the A_1-singularity.

Theorem 2.2 ([IU05]). *Conjecture 2.1 is true when* $n = 1$.

2.2 Bridgeland's stability conditions

Conjecture 2.1 is deeply related to the following conjecture proposed by Bridgeland.

Motivated by some physical ideas, Bridgeland considers the set of *stability conditions* $\operatorname{Stab}\mathcal{D}$ on a given triangulated category $\mathcal{D}$ ([Br02]). He also shows that $\operatorname{Stab}\mathcal{D}$ has a natural topology and that it is a complex manifold. We call $\operatorname{Stab}\mathcal{D}$ *the space of stability conditions* on $\mathcal{D}$.

Let us consider below the space of stability conditions on $D_Z(X)$, where X and Z are as in Theorem 1.4.

Conjecture 2.3 (cf. [Br03]). $\operatorname{Stab} D_Z(X)$ *is simply connected and connected.*

Our interesting observation is as follows.

Proposition 2.4 ([IU05]). *Assume that* $\operatorname{Stab} D_Z(X)$ *is connected. Then Conjecture 2.1 holds precisely when Conjecture 2.3 holds.*

Combining Theorem 2.2 with Proposition 2.4, we conclude that $\operatorname{Stab} D_Z(X)$ is simply connected when $\operatorname{Stab} D_Z(X)$ is connected and $n = 1$.

References

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