UNIVERSAL ABELIAN COVERS OF COMPLEX SURFACE SINGULARITIES

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We are interested in questions studied recently in joint work with Walter Neumann, relating the geometry to the topology of germs of complex normal (hence isolated) surface singularities $(Y, 0)$. For instance, one can consider isolated hypersurface singularities

$$\{f(x, y, z) = 0\} \subset \mathbb{C}^3,$$

an instructive example of which are the Brieskorn singularities

$$V(p, q, r) = \{(x, y, z) \in \mathbb{C}^3 |\ x^p + y^q + z^r = 0\}.$$

Intersecting one of these surfaces with a small 5-sphere about the singular point gives the *link* Σ of the singularity. In this situation, it is a compact oriented 3-manifold, and the cone over the pair (S^5, Σ) describes the local topology of the singular point in $\mathbb{C}^3$. One is in a similar situation for any normal surface singularity, except that the link now sits in a sphere of larger dimension; so we do not consider "knot invariants", which govern the embedding in spheres. (Recall that for irreducible plane curve singularities, the "link" itself is just a circle, and the interesting part is the knotting in S^3.) It is well-known that many geometric features of the singularity are not encoded in its link; for instance, the Brieskorn links $\Sigma(2, 7, 14)$ and $\Sigma(3, 4, 12)$ are diffeomorphic, but have many different invariants (e.g., multiplicity and Milnor number).

A good resolution $(\tilde{Y}, E) \to (Y, 0)$ is one for which the exceptional configuration consists of smooth projective curves intersecting transversally. The link can be viewed as the boundary of a tubular neighborhood of E in $\tilde{Y}$, and a deep theorem of W. Neumann shows that conversely the link determines the exceptional configuration (with some well-known exceptions).

Now, E is a tree of rational curves if and only if the first betti number of Σ is 0, i.e. is a $\mathbb{Q}HS$ or *rational homology sphere*. This is a common situation, e.g. for a Brieskorn with p prime to qr. A $\mathbb{Q}HS$ link has finite first homology group, so it has a (topological) universal abelian cover which is finite over it. This covering can be realized on the level of germs of singularities via a finite Galois map $(X, 0) \to (Y, 0)$, unramified off the singular points. We call $(X, 0)$ the *universal abelian cover*, or UAC, of our original singularity (of course, we also are including the group and its action in the definition). A purely algebraic definition of this cover may be given as well.

The main point of this discussion is that in many important cases, the UAC of $(Y, 0)$ is an explicitly given complete intersection intersection singularity, of so-called "splice type" , whose equations (up to higher order terms) and group action can be written down just from the topology of Σ. In other words, the topology of the link of $(Y, 0)$ produces nice equations *not* for the singularity itself, but for its universal abelian cover. Such singularities we call "splice quotients"; they are based on a new and combinatorially interesting class of singularities, called "complete intersections of splice type," generalizing Brieskorn complete intersections. Our main theorem states that if a link satisfies two fairly mild topological properties (the "semigroup and congruence conditions"), then it is the link of a splice quotient.

Our talk will explain all the constructions and conditions, motivating the result first by looking at the rational double points and the equations produced by Felix Klein. We describe the first general result in this direction, due to W. Neumann in the 1980's, describing weighted homogeneous singularities with $\mathbb{Q}HS$ link; their UAC's are Brieskorn complete intersections. Very recently, T. Okuma has proved the important result that every rational surface singularity satisfies the splice and semigroup condition, and is in fact always a splice quotient. Finally, we will describe a Conjecture (which hopefully soon will be a Theorem) which explains precisely when a given singularity is a splice quotient.

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