Rigid Cohen-Macaulay modules over a quotient singularities of dimension three

Yuji Yoshino
Math. Department, Faculty of Science
Okayama University, Okayama 700-8530, Japan
yoshino@math.okayama-u.ac.jp

This is partly a joint work with O. Iyama.

Let G be a finite subgroup of $\mathrm{GL}(3,\mathbb{C})$ which naturally acts on the formal power series ring $S = k[[x,y,z]]$. We denote by R the invariant subring of S by this action of G. We employ the following assumptions on the group G:

(i) Any element $\sigma \neq 1$ of G does not have eigen value 1.

(ii) G is a subgroup of $\mathrm{SL}(3,\mathbb{C})$.

It is easy to see that the conditions (i) and (ii) are equivalent to that R is a Gorenstein ring with isolated singularity. We are interested in the category of maximal Cohen-Macaulay modules $\mathrm{CM}(R)$ over R.

Under the assumptions, we know that the Auslander-Reiten-Serre duality holds for the Tate cohomologies, i.e.

$$\mathrm{Ext}^3_R(\check{\mathrm{E}}\mathrm{xt}^i_R(M,N),R) \cong \check{\mathrm{E}}\mathrm{xt}^{2-i}_R(N,M),$$

for any $i \in \mathbb{Z}$ and $M, N \in \mathrm{CM}(R)$. In other words, the stable category $\underline{\mathrm{CM}(R)}$ is a 2-Calabi-Yau category.

To state the first main theorem, we denote by Λ the skew-group ring of G over S, i.e. $\Lambda = \sum_{\sigma \in G} S\sigma$ with multiplication $(s\sigma)(t\tau) = s\sigma(t)\sigma\tau$ for $s,t \in S$ and $\sigma,\tau \in G$. Now let ρ be an idempotent $\frac{1}{|G|}\sum_{\sigma\in G}\sigma$ of Λ and consider the residue ring $\overline{\Lambda} = \Lambda/\Lambda\rho\Lambda$. Note that $\overline{\Lambda}$ is a finite R-algebra which is an artinian ring. We denote by $\overline{\Lambda}$-mod the category of finite left $\overline{\Lambda}$-modules.

Theorem 0.1 *There is a cateory equivalence between* $\mathrm{CM}(R)/[S]$ *and* $\overline{\Lambda}$-mod.

This theorem can be applied to several cases to classify particular modules. To give such an example, let

$$\sigma = \begin{pmatrix} \zeta & & \\ & \zeta & \\ & & \zeta \end{pmatrix},$$

where ζ is a primitive cubic root of unity, and let G be a cyclic group of order three which is generated by σ. In this case, the invariant subring R is (the completion of) the Veronese subring of degree three :

$$R = k[[\{\text{monomials of degree three in } x, y, z, \}]]$$

The action of G gives the G-graded structure on S in such a way that $S = S_0 \oplus S_1 \oplus S_2$, where each S_j is the R-module of semi-invariants that is defined as

$$S_j = \{f \in S \mid f^\sigma = \zeta^j f\}.$$

Note that $S_0 = R$. It is known that S_j $(j \in G)$ are maximal Cohen-Macaulay modules over R, and in particular they are reflexive R-modules of rank one, whose classes form the divisor class group of R. Particularly, $\{S_j\}$ are all of the maximal Cohen-Macaulay modules of rank one over R.

On the other hand, it follows from Theorem 0.1 that the category $\mathrm{CM}(R)$ of maximal Cohen-Macaulay modules over R is of wild representation type. Actually, $\overline{\Lambda}$-mod is nothing but the category of representations of the wild quiver :

$$Q = \left(\bullet \quad \begin{matrix} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{matrix} \quad \bullet \right)$$

Henceforth, we are interested in rigid maximal Cohen-Macaulay modules that is defined as follows:

Definition 0.2 An R-module M is called rigid if $\mathrm{Ext}^1_R(M, M) = 0$.

By computation, the modules $S_j (j \in G)$ and their any syzygies and any cosyzygies are rigid (and indecomposable). The second main theorem is the following:

Theorem 0.3 *Under such circumstances as above, let $\mathcal{S}$ be a sequence of indecomposable rigid maximal Cohen-Macaulay modules defined as follows:*

$$\mathcal{S} = (\cdots, \Omega^{-2}S_1,\ \Omega^{-2}S_2,\ \Omega^{-1}S_1,\ \Omega^{-1}S_2,\ S_1,\ S_2,\ \Omega^1 S_1,\ \Omega^1 S_2,\ \Omega^2 S_1,\ \Omega^2 S_2, \cdots).$$

Then any rigid maximal Cohen-Macaulay module over R is isomorphic to a module of the following form:

$$P^a \oplus Q^b \oplus R^c,$$

where a, b, c are nonnegative integers and $\{P, Q\}$ is a pair of two adjacent modules in the sequence $\mathcal{S}$.